§. Examples of A_p Weights

1. Power Functions

Q: For what real numbers a is the power function $|x|^a$ an A_p weight on $\mathbb{R}^n$?

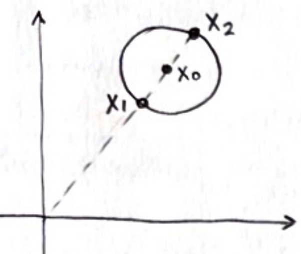
Look at supremum over balls:

$$\sup_B \left(\frac{1}{|B|}\int_B |x|^a dx\right)\left(\frac{1}{|B|}\int_B |x|^{-\frac{ap'}{p}} dx\right)^{p/p'}$$

Divide the balls in $\mathbb{R}^n$ into 2 categories:

Type I balls: radius$(B) < 3\cdot$dist$(B,0)$

Such balls do not contain the origin, so neither integral is affected.



Let $B = B(x_0, R)$ be such a ball.

$|x_2| = |x_0| + R < |x_0| + 3|x_1| \leq 4|x_0| \Rightarrow |x| \leq 4|x_0|, \forall x \in B$

$|x_1| = |x_0| - R > |x_0| - 3|x_1| \Rightarrow 4|x_1| > |x_0| \Rightarrow |x| \geq \frac{1}{4}|x_0|, \forall x \in B$

$\Rightarrow$ The quantity inside the supremum is

$$\lesssim |x_0|^a\left(|x_0|^{-a p'/p}\right)^{p/p'} = 1$$

Type II balls: All others: radius$(B) \geq 3$ dist$(B,0)$

Let $B(x_0,R)$ be such a ball. Then $|B| = v_n R^n$ (where v_n = volume of unit sphere in $\mathbb{R}^n$)

$\Rightarrow B(0,5R)$ has size comparable to B and contains it $\Rightarrow$ suffices to estimate over balls of the form $B(0,5R)$.

$5^n|B|$ $\quad$ obvious if $0 \in B$; if $0 \notin B$:

$|x_2| = 2R + |x_1| \leq 2R + \frac{R}{3} < 2R + \frac{R}{2}$

$\Rightarrow |x| < \frac{5R}{2}, \forall x \in B$

In that case the integrals become:

$$\left(\frac{1}{v_n(5R)^n}\int_{B(0,5R)}|x|^a dx\right)\left(\frac{1}{v_n(5R)^n}\int_{B(0,5R)}|x|^{-\frac{ap'}{p}}dx\right)^{p/p'}$$

$$=\left(\frac{n}{(5R)^n}\int_0^{5R} r^{a+n-1}dr\right)\left(\frac{n}{(5R)^n}\int_0^{5R} r^{-a\frac{p'}{p}+n-1}dr\right)^{p/p'}$$

$$\boxed{\int_0^k x^m dx = \frac{x^{m+1}}{m+1}\Big|_0^k \quad \text{finite} \iff \boxed{m+1>0}: \frac{k^{m+1}}{m+1}}$$

$$\text{Finite} \iff \begin{cases}(a+n-1)+1>0 \\ \left(-a\frac{p'}{p}+n-1\right)+1>0\end{cases} \qquad \begin{cases} a>-n \\ a\frac{p'}{p}<n;\ a<\frac{np}{p'}=n(p-1)\end{cases}$$

The integral becomes:

$$\frac{n}{(5R)^n}\,\frac{(5R)^{a+n}}{a+n}\left(\frac{n}{(5R)^n}\,\frac{(5R)^{-a\frac{p'}{p}+n}}{-a\frac{p'}{p}+n}\right)^{p/p'} = \frac{n}{a+n}\cancel{(5R)^a}\left(\frac{n}{-a\frac{p'}{p}+n}\right)^{p/p'}\cancel{(5R)^{-a}} \quad \text{(uniformly b'dd)}$$

$\Rightarrow$ **$|x|^a$ is an A_p weight in $\mathbb{R}^n$** $\iff \boxed{-n<a<n(p-1)}$

2. All bounded weights: $a \leq w(x) \leq b$, a.a. $x \in \mathbb{R}^n$

$$\Rightarrow \langle w\rangle_Q \underbrace{\langle w'\rangle_Q^{p-1}}_{\left\langle \frac{1}{w^{\frac{p'}{p}}}\right\rangle_Q^{p/p'}} \leq \frac{b}{a} \quad \forall Q.$$

3. A_p and BMO

→ There is a deep connection b/w A_p classes & BMO. Explore some facts about BMO:

$$\boxed{\|b\|_{BMO} := \sup_Q \frac{1}{|Q|}\int_Q |b(x) - \langle b\rangle_Q|\,dx}$$

John-Nirenberg Theorem: → proof next week

For all $b \in BMO(\mathbb{R}^n)$, all cubes Q_0, and all $\alpha > 0$:

$$\boxed{|\{x \in Q_0 : |b(x) - \langle b\rangle_{Q_0}| > \alpha\}| \le C|Q_0| e^{-A\alpha/\|b\|_{BMO}}}$$

with constants: $C = e^{2^n e}$ and $A = \frac{1}{2^n e}$.

Corollary 1: Every BMO function is exponentially integrable over any cube. Specifically, for any $\gamma < \frac{1}{2^n e}$, there is a constant $C_{n,\gamma}$ such that:

$$\boxed{\frac{1}{|Q|}\int_Q e^{\gamma|b(x)-\langle b\rangle_Q|/\|b\|_{BMO}}\,dx \le C_{n,\gamma}} \quad \forall b \in BMO(\mathbb{R}^n), \forall \text{ cube } Q.$$

Proof: $\frac{1}{|Q|}\int_Q e^{|f(x)|}\,dx = \frac{1}{|Q|}\int_0^\infty e^\alpha |\{x \in Q : |f(x)| > \alpha\}|\,d\alpha$

$$\Rightarrow \frac{1}{|Q|}\int_Q e^{\gamma|b(x)-\langle b\rangle_Q|/\|b\|_{BMO}}\,dx = \frac{1}{|Q|}\int_0^\infty e^\alpha \left|\left\{x \in Q : |b(x) - \langle b\rangle_Q| > \frac{\alpha\|b\|_{BMO}}{\gamma}\right\}\right| d\alpha$$

$$\le \frac{1}{|Q|}\int_0^\infty e^\alpha \cdot C|Q| e^{-A\alpha/\gamma}\,d\alpha = \underbrace{C\int_0^\infty e^{\alpha(1-A/\gamma)}\,d\alpha}_{C_{n,\gamma}}.$$

■

Corollary 2: For all $0 < p < \infty$, there exists a finite constant B_p such that:

$$\boxed{\sup_Q \left(\frac{1}{|Q|}\int_Q |b(x) - \langle b\rangle_Q|^p\,dx\right)^{1/p} \le B_p \|b\|_{BMO}}$$

Proof: $\frac{1}{|Q|}\int_Q |b(x)-\langle b\rangle_Q|^p\,dx = \frac{p}{|Q|}\int_0^\infty \alpha^{p-1}|\{x \in Q : |b(x) - \langle b\rangle_Q| > \alpha\}|\,d\alpha$

$$\le \frac{p}{|Q|} C|Q| \int_0^\infty \alpha^{p-1} e^{-A\alpha/\|b\|_{BMO}}\,d\alpha = pC\frac{\|b\|_{BMO}^p}{A^p}\Gamma(p)$$

$$\Rightarrow \boxed{B_p = \left(p\Gamma(p)\frac{C}{A^p}\right)^{1/p}} \qquad \Gamma(p) := \int_0^\infty x^{p-1}e^{-x}\,dx$$

■

Remark: For $p > 1$, the converse also holds:

$$\frac{1}{|Q|}\int_Q |b - \langle b\rangle_Q| \le \frac{1}{|Q|}\left(\int_Q |b-\langle b\rangle_Q|^p\right)^{1/p} |Q|^{1/p'} = \left(\frac{1}{|Q|}\int_Q |b - \langle b\rangle_Q|^p\right)^{1/p}$$

⇒ for $1 < p < \infty$, this is an equivalent definition of BMO norm:

$$\|b\|_{BMO} \simeq \sup_Q \left(\frac{1}{|Q|}\int_Q |b - \langle b\rangle_Q|^p\right)^{1/p}, \quad \forall 1 < p < \infty.$$

~ All A_2 weights w are of the form $e^{b\eta}$, where $b \in BMO$!

Proposition: Let w be a weight on $\mathbb{R}^n$. Then $\ln(w) \in BMO$ if and only if there is $\eta > 0$ such that $w^\eta \in A_2$.

Proof: Suppose first that $\ln(w) \in BMO$. Then (exponential integrability):

$\forall \gamma < \frac{1}{2^n e}$, $\exists C_{n,\gamma}$ s.t.

$$\frac{1}{|Q|}\int_Q e^{\gamma |\ln(w(x)) - \langle \ln w \rangle_Q| / \|\ln(w)\|_{BMO}} dx \leq C_{n,\gamma}$$

Integrand: $e^{\pm \eta (\ln w(x) - \langle \ln w \rangle_Q)}$, where $\eta = \frac{\gamma}{\|\ln(w)\|_{BMO}}$

$$\Rightarrow \frac{1}{|Q|}\int_Q e^{\eta(\ln w(x) - \langle \ln w \rangle_Q)} dx \leq C_{n,\gamma} \quad \Big| \quad \frac{1}{|Q|}\int_Q e^{-\eta(\ln w(x) - \langle \ln w \rangle_Q)} dx \leq C_{n,\gamma}$$

$$\left(\frac{1}{|Q|}\int_Q w^\eta\right) e^{-\eta \langle \ln w \rangle_Q} \leq C_{n,\gamma} \quad \Big| \quad \left(\frac{1}{|Q|}\int_Q \frac{1}{w^\eta}\right) e^{\eta \langle \ln w \rangle_Q} \leq C_{n,\gamma}$$

$\Rightarrow$ multiply the estimates: $\langle w^\eta \rangle_Q \langle w^{-\eta} \rangle_Q \leq C_{n,\gamma}^2$, $\forall Q$ cube $\Rightarrow$ $\boxed{w^\eta \in A_2}$

Conversely: Say $w^\eta \in A_2$

$$\Rightarrow \frac{1}{|Q|}\int_Q e^{\eta |\ln w(x) - \langle \ln w \rangle_Q|} dx \leq \underbrace{\frac{1}{|Q|}\int_Q e^{\eta(\ln w(x) - \langle \ln w \rangle_Q)} dx}_{} + \underbrace{\frac{1}{|Q|}\int_Q e^{-\eta(\ln w(x) - \langle \ln w \rangle_Q)} dx}_{}$$

$$\begin{aligned} &\langle w^\eta \rangle_Q e^{-\eta \langle \ln w \rangle_Q} \\ &= \langle w^\eta \rangle_Q e^{-\eta \frac{1}{|Q|}\int_Q \ln w} \\ &= \langle w^\eta \rangle_Q e^{\frac{1}{|Q|}\int_Q \ln(\frac{1}{w^\eta})} \\ &\leq \langle w^\eta \rangle_Q e^{\ln \langle w^{-\eta} \rangle_Q} \\ &= \langle w^\eta \rangle_Q \langle w^{-\eta} \rangle_Q \\ &\leq [w^\eta]_{A_2} \end{aligned}$$

Jensen:

$$\frac{1}{|Q|}\int_Q \ln\left(\frac{1}{w^\eta}\right) \leq \ln\left(\frac{1}{|Q|}\int_Q \frac{1}{w^\eta}\right)$$

$$\begin{aligned} &\langle w^{-\eta} \rangle_Q e^{\eta \langle \ln w \rangle_Q} \\ &= \langle w^{-\eta} \rangle_Q e^{\frac{1}{|Q|}\int_Q \ln(w^\eta)} \\ &\leq \langle w^{-\eta} \rangle_Q e^{\ln \langle w^\eta \rangle_Q} \\ &= \langle w^{-\eta} \rangle_Q \langle w^\eta \rangle_Q \\ &\leq [w^\eta]_{A_2} \end{aligned}$$

$\Rightarrow \ln(w) \in BMO$.

Corollary: Let w be a weight and suppose $w^\eta \in A_p$ for some $\eta > 0$, $1 < p < \infty$. Then $\ln(w) \in BMO$.

Proof: If $1 < p \leq 2$, then $w^\eta \in A_p \subset A_2 \Rightarrow$ result follows.

If $p > 2$, then

$$w^{-\frac{\eta}{p-1}} \in A_{p'}, \text{ where } p' < 2 \Rightarrow (w')^\eta \in A_2$$

$$\Rightarrow \underset{-\frac{1}{p-1}\ln(w)}{\underset{\parallel}{\ln(w')}} \in BMO \Rightarrow \ln(w) \in BMO.$$

9. Sparse & Carleson Families:

Let $\mathcal{D}$ be a dyadic lattice on $\mathbb{R}^n$.

DEF.: Let $0<\eta<1$. A collection $\mathcal{S}\subset\mathcal{D}$ is called η-sparse if for every $Q\in\mathcal{S}$ there is a measurable subset $E_Q\subset Q$ such that the sets $\{E_Q\}_{Q\in\mathcal{S}}$ are pairwise disjoint and

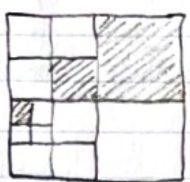
$$\boxed{|E_Q|\geq\eta|Q|,\ \forall Q\in\mathcal{S}} \Rightarrow |Q\setminus E_Q|\leq(1-\eta)|Q|.$$

DEF.: Let $\Lambda>1$. A family of cubes $\mathcal{S}\subset\mathcal{D}$ is called Λ-Carleson if:

$$\boxed{\sum_{\substack{P\in\mathcal{S}\\ P\subset Q}}|P|\leq\Lambda|Q|,\ \forall Q\in\mathcal{D}}$$

Remark: It suffices to verify the Carleson condition on all $Q\in\mathcal{S}$:

$$\boxed{\sum_{\substack{P\in\mathcal{S}\\ P\subset Q}}|P|\leq\Lambda|Q|,\ \forall Q\in\mathcal{S}} \Rightarrow \boxed{\sum_{\substack{P\in\mathcal{S}\\ P\subset Q}}|P|\leq\Lambda|Q|,\ \forall Q\in\mathcal{D}}$$



$Q\in\mathcal{D}$

Let $Q\in\mathcal{D}$. If Q contains no elements of $\mathcal{S}$, we are done (LHS is 0). If $Q\in\mathcal{S}$ we are also done, so assume $Q\notin\mathcal{S}$ but Q contains some elements of $\mathcal{S}$. Let $\boxed{ch_{\mathcal{S}}Q}$ (the "$\mathcal{S}$-children" of Q) be the maximal elements of $\mathcal{S}$ that are strictly contained in Q. Then:

$$\sum_{\substack{P\in\mathcal{S}\\ P\subset Q}}|P| = \sum_{Q'\in ch_{\mathcal{S}}Q}\underbrace{\sum_{\substack{P\in\mathcal{S}\\ P\subset Q'}}|P|}_{\leq\Lambda|Q'|\text{ b/c }Q'\in\mathcal{S}} \leq \Lambda\sum_{Q'\in ch_{\mathcal{S}}Q}|Q'|\leq\Lambda|Q| \quad \text{b/c the } Q'\in ch_{\mathcal{S}}Q \text{ are disjoint.}$$

Remarkable Property: The sparseness & Carleson conditions are equivalent:

$\boxed{\eta\text{-sparse}} \Rightarrow \boxed{\tfrac{1}{\eta}\text{-Carleson}}$ (easy)

Let $Q\in\mathcal{S}$. Then:

$$\sum_{\substack{P\in\mathcal{S}\\ P\subset Q}}|P| \leq \sum_{\substack{P\in\mathcal{S}\\ P\subset Q}}\frac{1}{\eta}|E_P| = \frac{1}{\eta}\Big|\underbrace{\bigcup_{\substack{P\in\mathcal{S}\\ P\subset Q}}E_P}_{\text{b/c the sets }E_P\text{ are disjoint}}\Big| \leq \frac{1}{\eta}|Q|.$$

$\boxed{\Lambda\text{-Carleson}} \Rightarrow \boxed{\tfrac{1}{\Lambda}\text{-sparse}}$ (difficult).

$\hookrightarrow\ |Q|\leq\frac{1}{\eta}|E_Q|=\Lambda|E_Q|$

Remark: The sparseness property is something that can be readily used when working w/systems of cubes that are already known to be sparse, while the Carleson property is something that can be easily verified in many cases where the sparseness condition is not obvious at all.

For example, it is easy to see that:

The union of N Carleson systems w/constants $\Lambda_1, \dots, \Lambda_N$ is a Carleson system w/constant $\Lambda_1 + \dots + \Lambda_N$.

while to see directly that

The union of N sparse collections w/constants $\eta_1, \dots, \eta_N$ is a sparse system w/constant $\left(\frac{1}{\eta_1} + \dots + \frac{1}{\eta_N}\right)^{-1}$.

is next to impossible.

Check: Let $\mathcal{S}_1, \dots, \mathcal{S}_N$ be Carleson systems s.t. each $\mathcal{S}_j$ is Λ_j-Carleson. Let $\mathcal{S} := \bigcup_{j=1}^{N} \mathcal{S}_j$. Then for any $Q \in \mathcal{S}$:

$$\sum_{\substack{P \in \mathcal{S} \\ P \subset Q}} |P| \leq \sum_{j=1}^{N} \underbrace{\sum_{\substack{P \in \mathcal{S}_j \\ P \subset Q}} |P|}_{\leq \Lambda_j |Q|} \leq \sum_{j=1}^{N} \Lambda_j |Q| = (\Lambda_1 + \dots + \Lambda_N)|Q| \Rightarrow \mathcal{S} \text{ is } (\Lambda_1 + \dots + \Lambda_N)\text{-Carl.}$$

(regardless if $Q \in \mathcal{S}_j$ or not! Remember this holds for all $Q \in \mathcal{D}$).

$\Rightarrow$ The equivalent statement: $\{\mathcal{S}_j\}_{j=1}^{N}$ are each η_j-sparse $\Rightarrow$ each is $\frac{1}{\eta_j}$-Carleson $\Rightarrow$ their union $\mathcal{S}$ is $\left(\frac{1}{\eta_1} + \dots + \frac{1}{\eta_N}\right)$-Carleson $\Rightarrow \mathcal{S}$ is $\left(\frac{1}{\eta_1} + \dots + \frac{1}{\eta_N}\right)^{-1}$-sparse. ▤

Remark: The $\mathcal{S}$-children definition (a special case of sparse): (appears frequently in practice).

Suppose a family $\mathcal{S} \subset \mathcal{D}$ has the property:

$$\boxed{\sum_{P \in ch_{\mathcal{S}}(Q)} |P| \leq \alpha |Q|, \ \forall Q \in \mathcal{S}}$$

for some $\alpha \in (0,1)$. Then $\mathcal{S}$ is $(1-\alpha)$-sparse (and therefore $\frac{1}{1-\alpha}$-Carleson).

Pf.: Let the sets $\boxed{E_Q := Q \setminus \bigcup_{P \in ch_{\mathcal{S}} Q} P}$ for every $Q \in \mathcal{S}$. Then clearly $\{E_Q\}_{Q \in \mathcal{S}}$ are disjoint, and:

$$|E_Q| = \Big| Q \setminus \bigcup_{P \in ch_{\mathcal{S}} Q} P \Big| = |Q| - \Big| \bigcup_{P \in ch_{\mathcal{S}} Q} P \Big| = |Q| - \sum_{P \in ch_{\mathcal{S}} Q} |P| \geq |Q| - \alpha |Q| = (1-\alpha)|Q|$$

(by disjointness of $P \in ch_{\mathcal{S}} Q$)

$\Rightarrow |E_Q| \geq (1-\alpha)|Q| \Rightarrow \mathcal{S}$ is $(1-\alpha)$-sparse.

We can also prove directly that it is a Carleson collection: for some $Q \in \mathcal{S}$:

$$\sum_{\substack{P \in \mathcal{S} \\ P \subset Q}} |P| = |Q| + \sum_{P \in ch_{\mathcal{S}} Q} \sum_{\substack{P' \in \mathcal{S} \\ P' \subset P}} |P'| = |Q| + \sum_{P \in ch_{\mathcal{S}} Q} |P| + \sum_{P \in ch_{\mathcal{S}} Q} \sum_{P' \in ch_{\mathcal{S}} P} |P'| + \ldots$$

$$\leq |Q| + \alpha |Q| + \underbrace{\sum_{P \in ch_{\mathcal{S}} Q} \alpha |P|}_{\leq \alpha^2 |Q|} + \ldots$$

$$\leq |Q| (1 + \alpha + \alpha^2 + \ldots) = |Q| \frac{1}{1-\alpha} \text{ b/c } \alpha \in (0,1) \Rightarrow \mathcal{S} \text{ is } \frac{1}{1-\alpha}\text{-Carleson.}$$

∎